Superconformal Ward Identities and $N = 2$ Yang-Mills Theory

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Abstract

A reformulation of the superconformal Ward identities that combines all the superconformal currents and the associated parameters in one multiplet is given for theories with rigid $N = 1$ or $N = 2$ supersymmetry. This form of the Ward Identities is applied to spontaneously broken $N = 2$ Yang-Mills theory and used to derive a condition on the low energy effective action. This condition is satisfied by the solution proposed by Seiberg and Witten.

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0. Introduction

Some tine ago it was shown that certain four-dimensional supersymmetric theories are quantum mechanically superconformally invariant or, equivalently, finite. These theories are: $N = 4$ Yang-Mills theory [1], a large class of $N = 2$ Yang-Mills theories coupled to $N = 2$ matter [2] and certain $N = 1$ theories [3]. It was also shown that in perturbation theory the $N = 2$ Yang-Mills beta-function has only one-loop contributions [2,4]. Although the significance of these results was not immediately apparent, a number of other developments have taken place which have focused attention on these theories and which have relied upon their conformal properties. In particular, the electro-magnetic duality conjecture of Montonen and Olive [5] is believed to be most likely [6] to be valid in these superconformally invariant theories as the couplings do not run under a change of scale and so any symmetry that inverts the coupling makes sense at all scales [7]. It has also been suggested that there may be further examples of $N = 1$ theories [24] and $N = 2$ theories [8] which have non-trivial fixed points.

Recently [9], it has been found that one can determine the low energy effective action of spontaneously broken $N = 2$ Yang-Mills theory. In practice this means that part of the effective action which is a chiral sub-integral depending holomorphically on the field strength superfield, A, of the unbroken $U(1)$. Of course, this is only a part of the full effective action (for the $U(1)$ fields) which is an integral over all of superspace of a function of A and $\bar{A}$ and derivatives and which in general has non-local contributions. The determination of the low-energy effective action in [9] makes essential use of properties of the theory related to electromagnetic duality.

An at first sight unrelated development was the work of BPZ [10], who solved a large class of conformal invariant theories in two dimensions. This represented the first systematic non-perturbative solution of a class of quantum field theories. The most likely theories in four dimensions for which one might try to emulate this achievement are the theories which have extended supersymmetry. An early signal that such progress may be possible

was the discovery that at a fixed point the anomalous dimension of a chiral superfield was fixed in terms of its weight under R-symmetry transformations [11].

Recently [12], the authors have investigated the consequences of superconformal invariance for certain Green's functions in rigidly supersymmetric theories in four spacetime dimensions. It has been argued [12] that in superconformal theories one can solve for the Green's functions in the chiral or analytic sectors. Chiral sectors occur in $N = 1$ matter and Yang-Mills theories and in $N = 2$ Yang-Mills theories and this result generalises the analogous result [13] in two dimensions. Analytic sectors occur in $N = 2$ matter (coupled to Yang-Mills) and $N = 4$ Yang-Mills theory. Indeed, in the latter theory a large class of Green's functions are analytic and so the theory is at least partially soluble.

Although $N = 2$ Yang-Mills theory is not superconformally invariant there are still Ward Identities corresponding to superconformal transformations which have appropriate anomalous contributions. In this paper we derive the anomalous superconformal Ward identity and find the conditions that it implies for the low energy effective action of the spontaneously broken theory. The latter has the form $\int d^4x \int d^4\theta F(A) + c.c$, and we show that the superconformal Ward identity implies that F satisfies

$$a\frac{\partial F}{\partial a} - 2F = 8\pi i\beta_1 u.$$

for any gauge group, where β_1 is the coefficient of the one loop beta function and $a = < A >$. This constraint is indeed satisfied by the solution for F found by Seiberg and Witten. This was first shown for the case of gauge group $SU(2)$ spontaneously broken to $U(1)$ in reference [14] and for the $SU(N)$, $SO(N)$ and $Sp(N)$ gauge groups spontaneously broken to their Cartan subalgebras in the presence of certain $N = 2$ matter in references [15,16]. In these papers [14,15,16], the above condition has been observed to hold phenomologically, that is, the authors have assumed the Seiberg-Witten solution with its associated hyperelliptic curve and have derived the above simple condition from it. In this paper, for the first time this condition, and therefore part of the information about the low energy effective action, will be derived directly from the underlying field theory without assuming electromagnetic

duality.

In order to derive this result, we give a new superspace formulation of the supercon-
formal currents and their corresponding Ward identities. It is well known [17] that the
superymmetry and internal symmetry currents and the energy-momentum tensor belong
to the supercurrents $J_{\alpha\dot\alpha}$ and J for $N = 1$ and $N = 2$ respectively. In a superconformal
theory, moreover, moments of the supersymmetry currents and the energy-momentum
tensor are also conserved. In section one, we construct a supermultiplet of moments by
combining all the superconformal currents and their parameters into a single superfield.
For $N = 1$ theories this superfield is $f^{\alpha\dot\alpha} J_{\alpha\dot\alpha}$ where $f^{\alpha\dot\alpha}$ is a superfield that contains the
superconformal parameters. A more complicated expression is given for the case of $N = 2$
supersymmetry.

In sections two and three the form of the superconformal Ward identities are derived
for rigid supersymmetric $N = 1$ and $N = 2$ theories respectively. This is first done for
the Ward identities that involve the supercurrents $J_{\alpha\dot\alpha}$ and J and then for the form that
involves all the superconformal currents and the associated parameters. The advantage of
this latter formulation is that it allows one to deal with all the superconformal symmetries
together in a simple way.

In section four, we apply the superconformal Ward identities to the spontaneously broken
$N = 2$ Yang-Mills theory and derive a superfield constraint on the Seiberg-Witten solution.
We find that one special case of this equation is the constraint equation given above.
We then derive an alternative form of this equation by replacing the anomaly part by a
derivative with respect to a conformal compensating field.

1. Currents in Supersymmetric Theories

In this section we review the currents that appear in $N = 1$ and 2 supersymmetric

theories. We begin by considering a current corresponding to an internal symmetry. In $N = 1$ supersymmetric theories such an internal symmetry current belongs to a real linear supermultiplet L and the conservation condition is $\bar{D}^2 L = 0$. For an $N = 2$ supersymmetric theory, on the other hand, an internal symmetry current is a component of a real symmetric superfield, L_{ij}, which transforms under the triplet representation of $SU(2)$. The current conservation condition is $D_{\alpha(i} L_{jk)} = 0$. For both of these multiplets the internal (spacetime) currents occur at the $\theta\bar{\theta}$ level and so the superfields L and L_{ij} have dimension 2. We observe that an internal symmetry current has superpartners that have no interpretation as conserved currents in spacetime.

Let us now consider the supersymmetry currents, $\{j_{\alpha i}\}$, $i = 1, \ldots, N$, and the supermultiplets they belong to. The supersymmetry variation of $j_{\alpha i}$ has the form $\delta j_{\alpha i} \sim [j_{\alpha i}, \epsilon^{\beta j} Q_{\beta j}]$ and must, when integrated, give the correct relation for the anti-commutators of two supersymmetry charges that occur in the supersymmetry algebra. We therefore conclude that the supersymmetry current is in the same multiplet as the energy-momentum tensor [17]. By extending this argument we find the supercurrent multiplet corresponding to superconformal symmetry also includes the internal symmetry currents whose charges appear in the supersymmetry algebra. For $N = 1$ there is only one such current, the R current, while for $N = 2$ there is the R current as well as the currents corresponding to the internal $SU(2)$ symmetry. Proceeding in this manner one can systematically construct the supercurrent multiplet associated with superconformal symmetry in a purely algebraic fashion [30]. In addition to the conserved currents listed above one also finds that the components of the supercurrent include certain moments of the energy-momentum tensor and of the supersymmetry currents.

The $N = 1$ supercurrent is described by a real superfield $J_{\alpha\dot{\alpha}}$. It has dimension 3 and obeys, in the absence of any anomalies, the conservation condition

$$\bar{D}^{\dot{\alpha}} J_{\alpha\dot{\alpha}} = 0. \tag{1.1}$$

For $N = 2$ the supersymmetry current belongs to a real scalar superfield J which has

dimension 2 and which, again in the absence of anomalies, obeys the equation

$$D_{ij}J = 0. \tag{1.2}$$

where

$$D_{ij} := D_{\alpha i}D_j^\alpha. \tag{1.3}$$

For both $N = 1$ and $N = 2$ superconformal theories, the parameters associated with superconformal transformations can be combined in a superfield of parameters, $f^{\alpha\dot\alpha}$, which is subject to the constraint

$$D_{(\alpha i}f_{\beta)\dot\beta} = 0. \tag{1.4}$$

Such a parameter determines a (real) superconformal Killing vector field X_f given by

$$X_f = f^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha} + \phi^{\alpha i}\partial_{\alpha i} - \bar\phi_i^{\dot\alpha}\bar D_{\dot\alpha}^i \tag{1.5}$$

where

$$\phi^{\alpha i} := -\frac{i}{2}\bar D_{\dot\alpha i}f^{\alpha\dot\alpha}. \tag{1.6}$$

By definition, a superconformal Killing vector field is one which preserves chirality in super Minkowski space, i.e. one which satisfies

$$[\bar D_{\dot\alpha i}, X_f] \sim \bar D_{\dot\alpha i}. \tag{1.7}$$

The components of $f^{\alpha\dot\alpha}$ can be defined to be given by the following superfields evaluated at $\theta = 0$:

$$\begin{aligned}
v^{\alpha\dot\alpha} &:= f^{\alpha\dot\alpha}|, \\
\bar\eta_i^{\dot\alpha} &:= \frac{i}{2}D_{\alpha i}f^{\alpha\dot\alpha}|, \\
\eta^{\alpha i} &:= -\frac{i}{2}\bar D_{\dot\alpha}^i f^{\alpha\dot\alpha}|, \\
w_{ij} &:= [D_{\alpha i}, \bar D_{\dot\alpha j}]f^{\alpha\dot\alpha}|,
\end{aligned} \tag{1.8}$$

where the vertical bar denotes evaluation of a superfield at $\theta = 0$. It follows from equation (1.3) that the fields $v^{\alpha\dot\alpha}$, $\eta^{\alpha i}$ and its complex conjugate obey the conformal Killing equation and the spinor Killing equation respectively. That is,

$$\partial_{(\alpha(\dot\alpha}v_{\beta)\dot\beta)} = 0 \tag{1.9}$$

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and

$$\partial_{(\alpha|\dot{\alpha}|}\eta^i_{\beta)} = 0. \tag{1.10}$$

Solving equations (1.9) and (1.10), one sees that $v^{\alpha\dot{\alpha}}$ contains translations, dilations and special conformal transformations, while $\eta^{\alpha i}$ and its complex conjugate contain ordinary (Q) and special (S) supersymmetry transformations. The x-independent field w_{ij} is the parameter of internal symmetry transformations of the supersymmetry algebra.

In a conformal field theory, the energy-momentum tensor $T_{\mu\nu}$, assumed to be symmetric, traceless and conserved, can be combined with the parameter of conformal transformations, f^μ, into a conserved current J^f_μ, $J^f_\mu := f^\nu T_{\mu\nu}$, the conservation of J^f_μ being a consequence of the constraints on $T_{\mu\nu}$ and the fact that f^μ is a conformal Killing vector,

$$\partial_{(\mu} f_{\nu)} = \frac{2}{d}\eta_{\mu\nu}\partial_\rho f^\rho, \tag{1.11}$$

where d is the dimension of spacetime. Similarly, in an $N = 1$ superconformal field theory, the currents and their associated parameters can be neatly packaged into a real superfield $J^f = f^{\alpha\dot{\alpha}}J_{\alpha\dot{\alpha}}$. We observe that if $J_{\alpha\dot{\alpha}}$ is not anomalous then J^f is a linear multiplet, i.e.

$$\bar{D}^2 J^f = 0. \tag{1.12}$$

We note that the superconformal currents and their parameters when packaged in J^f have the same dimension and obey the same conservation equation as an internal symmetry current L which was discussed at the beginning of the section.

In an $N = 2$ superconformal theory we may likewise combine all the superconformal currents and their associated parameters into a real superfield J^f_{ij} which is symmetric in its $SU(2)$ indices i, j. This superfield is

$$J^f_{ij} = i\left(f^{\alpha\dot{\alpha}}D_{\alpha i}\bar{D}_{\dot{\alpha}j}J + \frac{1}{2}(D_{\alpha i}f^{\alpha\dot{\alpha}})\bar{D}_{\dot{\alpha}j}J - \frac{1}{2}(\bar{D}_{\dot{\alpha}i}f^{\alpha\dot{\alpha}})D_{\alpha j}J + \frac{1}{4}(D_{\alpha i}\bar{D}_{\dot{\alpha}j}f^{\alpha\dot{\alpha}})J\right) \tag{1.13}$$

The coefficients in the above expression are determined by the requirement that

$$D_{\alpha(i}J^f_{jk)} = 0, \tag{1.14}$$

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if the supercurrent J is conserved, that is, satisfies equation (1.2). Thus the $N = 2$ superconformal currents follow the same pattern as the $N = 1$ superconformal currents; namely the superconformal currents and their parameters combine into a superfield which is of the same type as an internal symmetry current.

In this paper, we want to consider theories which are classically superconformal invariant, but which develop anomalies quantum mechanically. The theories of interest to us are invariant under Poincaré and Q-supersymmetry transformations, but have anomalies in some, or all, of the remaining superconformal symmetries. In the presence of anomalies the supercurrent no longer obeys equation (1.1) or (1.2) which become modified. Although there are several possible types of anomaly for $N = 1$ supersymmetric theories, we will assume that the supercurrent $J_{\alpha\dot\alpha}$ obeys the operator equation

$$\bar{D}^{\dot\alpha} J_{\alpha\dot\alpha} = D_\alpha S \tag{1.15}$$

where S is a chiral superfield ($\bar{D}_{\dot\alpha} S = 0$). For $N = 2$ supersymmetric Yang-Mills theories, we will show that the anomaly is of the form

$$D_{ij} J = \bar{D}_{ij} \bar{S} \tag{1.16}$$

where S is a chiral superfield, $\bar{D}_{\dot\alpha i} S = 0$.

Anomalies in the supercurrent modify the conservation equations (1.12) and (1.13). It is straightforward to verify that if $J_{\alpha\dot\alpha}$ has the conformal anomaly of equation (1.15) then

$$\bar{D}^2 J^f = -\bar{D}_{\dot\alpha} f^{\alpha\dot\alpha} D_\alpha S + i f^{\alpha\dot\alpha} \partial_{\alpha\dot\alpha} S, \tag{1.17}$$

where $D^2 = \frac{1}{2} D_\alpha D^\alpha$, $\bar{D}^2 = -\frac{1}{2} \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha}$. The corresponding equation for $N = 2$ implies that J_{ij} obeys the equation

$$D_{\alpha(i} J^f_{jk)} = -\frac{i}{4} \bar{D}_{\dot\alpha(i} f_\alpha{}^{\dot\alpha} D_{jk)} J = -\frac{i}{4} \bar{D}_{(ij|} (\bar{D}_{\dot\alpha|k)} f_\alpha{}^{\dot\alpha} \bar{S}). \tag{1.18}$$

2. N=1 Supersymmetric Ward Identities

Let us consider an $N = 1$ supersymmetric theory which contains a set of chiral superfields collectively denoted by φ, their conjugates $\bar{\varphi}$ and the Yang-Mills potential V. The most general renormalizable action is of the form

$$\int d^4x d^4\theta \, \bar{\varphi} e^{gV} \varphi + Im\{\frac{\tau}{4\pi} \int d^4x d^2\theta \, \text{tr}(W_\alpha W^\alpha)\} + \int d^4x d^2\theta \, U(\varphi) + \text{c.c.}. \qquad (2.1)$$

where U is the superpotential which is at most cubic in the chiral superfields and $\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$.

We denote the effective action of this theory by Γ. We wish to consider the constraints placed on Γ by superconformal symmetry, that is, by the superconformal Ward identity. This subject has been studied extensively in the past, see for example references [18,19] and references therein. Here we will give a simple derivation of the WI based on a superspace version of Noether's identification of the current. We shall ignore the complications which arise due to gauge-fixing and ghosts as these will play no part in the applications in this paper.

We therefore consider the variation of Γ under the superspace analogue of (infinitesimal) local reparametrizations. The parameter of these transformations is a spinor superfield L^α. The chiral superfield φ, which obeys a flat space chiral condition and has R weight q, transforms under L^α transformations as

$$\delta\varphi = -\bar{D}^{\dot\alpha} L^\alpha \partial_{\alpha\dot\alpha} \varphi - i\bar{D}^2 L^\alpha D_\alpha \varphi + q\triangle\varphi$$
$$= -i\bar{D}^2(L^\alpha D_\alpha \varphi - qD_\alpha L^\alpha \varphi), \qquad (2.2)$$

where

$$\triangle = -\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha} L^\alpha + iD_\alpha \bar{D}^2 L^\alpha. \qquad (2.3)$$

The Yang-Mills potential transforms as

$$\delta V = -i(\bar{D}^2 L^\alpha)D_\alpha V - \frac{1}{2}\bar{D}^{\dot\alpha} L^\alpha \partial_{\dot\alpha\alpha} V$$
$$= -i\bar{D}^2(L^\alpha D_\alpha V) + \frac{i}{2}\bar{D}^{\dot\alpha} L^\alpha [D_\alpha, D_{\dot\alpha}]V + iL^\alpha W_\alpha + \text{c.c.} \qquad (2.4)$$

By definition, the supercurrent $J_{\alpha\dot{\alpha}}$ couples to the supergravity superfield $H^{\alpha\dot{\alpha}}$ at linearised order in $H^{\alpha\dot{\alpha}}$ in the form

$$2 \int d^4x d^4\theta \ H^{\alpha\dot{\alpha}} J_{\alpha\dot{\alpha}} \tag{2.5}$$

Since the transformation of $H^{\alpha\dot{\alpha}}$ is given by

$$\delta H^{\alpha\dot{\alpha}} = -\frac{i}{2}(D^\alpha \bar{L}^{\dot{\alpha}} + \bar{D}^{\dot{\alpha}} L^\alpha) \tag{2.6}$$

the variation of (2.5) contributes

$$-i \int d^4x d^4\theta \ \bar{D}^{\dot{\alpha}} L^\alpha J_{\alpha\dot{\alpha}} + \text{c.c.} \tag{2.7}$$

to zeroth order in $H^{\alpha\dot{\alpha}}$. This variation must be cancelled by the variation of Γ under the transformation of equations (2.2) and (2.4) and we will regard this variation as the definition of the supercurrent associated with Γ. To be precise, we take the supercurrent to be defined by

$$\delta\Gamma = i \int d^4x d^4\theta \ (\bar{D}^{\dot{\alpha}} L^\alpha) J_{\alpha\dot{\alpha}} + \text{c.c.} \tag{2.8}$$

We now find an expression for $J_{\alpha\dot{\alpha}}$ using the method explained above. From equations (2.2) and (2.4) we find that

$$\delta\Gamma = \int d^4x d^4\theta \Big\{ \{L^\alpha D_\alpha \varphi - q D_\alpha L^\alpha \varphi\} \frac{\delta\Gamma}{\delta V}$$
$$+ (-i\bar{D}^2(L^\alpha D_\alpha V) + \frac{i}{2}\bar{D}^{\dot{\alpha}} L^\alpha [D_\alpha, D_{\dot{\alpha}}]V + iL^\alpha W_\alpha)\frac{\delta\Gamma}{\delta V} \Big\} + c.c \tag{2.9}$$

where q is the R weight of φ which must be $\frac{1}{3}$ if we have one chiral superfield with a cubic superpotential. If we restrict our attention to a $U(1)$ gauge theory then gauge invariance implies that

$$\bar{D}^2 \frac{\delta\Gamma}{\delta V} = 0 \tag{2.10}$$

and so we can discard the first term in the second bracket above. In what follows we will consider the Abelian theory, but the modification to the non-Abelian case can be made.

Having identified the supercurrent $J_{\alpha\dot{\alpha}}$ in terms of the variations, it is straightforward to write down the Ward identity. For a superconformal theory, there are no anomalies and

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the Ward identity is given by setting the $\delta\Gamma$ of equation (2.8) equal to that of equation (2.9). For theories with superconformal anomalies the supercurrent $J_{\alpha\dot{\alpha}}$ will obey the operator equation (1.15) which is valid in Green's functions. In recovering the Ward identity from this operator equation we find additional terms which arise due to the fact that the derivatives are outside the time ordering of the Green's function so that they act on the time ordering as well as the current. These extra (contact) terms are none other than the variations of the fields, i.e. the $\delta\Gamma$ of equation (2.9).

Taking the anomaly into account we therefore find that the Ward Identity for an $N = 1$ rigid supersymmetric theory for the above transformations is given by

$$\int d^4x d^4\theta \Big\{ (L^\alpha D_\alpha \varphi - q D_\alpha L^\alpha \varphi) \frac{\delta\Gamma}{\delta\varphi} \Big\} + (iL^\alpha W_\alpha) \frac{\delta\Gamma}{\delta V} - i(\bar{D}^{\dot{\alpha}} L^\alpha J_{\alpha\dot{\alpha}}) \cdot \Gamma + \text{c.c.}$$

$$= +i \int d^4x d^4\theta (L^\alpha D_\alpha S) \cdot \Gamma + \text{c.c.} \qquad (2.11)$$

In this equation we have redefined the supercurrent by

$$J_{\alpha\dot{\alpha}} \rightarrow J_{\alpha\dot{\alpha}} - \frac{1}{2}[D_\alpha, D_{\dot{\alpha}}]V \frac{\delta\Gamma}{\delta V}, \qquad (2.12)$$

in order to obtain a supercurrent which is gauge invariant. The non-gauge invariance of the supercurrent is an artifact of the way we have derived it. Essentially, we have used the coupling of the theory to supergravity so that the superfields and their variations can involve spinorial covariant derivatives that contain the supergravity superfield $H^{\alpha\dot{\alpha}}$. We have chosen the chiral superfield to obey a flat-space chirality condition; however, the Abelian gauge invariance is realised with a chiral parameter Λ that satisfies $\bar{\mathbf{D}}_{\dot{\alpha}}\Lambda = 0$ where $\bar{\mathbf{D}}_{\dot{\alpha}}$ is the spinorial covariant derivative which involves $H^{\alpha\dot{\alpha}}$. The flat space chiral parameter Λ_0, satisfying $\bar{D}_{\dot{\alpha}}\Lambda_0 = 0$, is related to Λ by

$$\Lambda = \Lambda_0 - \frac{i}{2} H^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \Lambda_0 + \dots. \qquad (2.13)$$

Gauge invariance of the effective action plus the lowest order supergravity coupling of equation (2.5) implies that

$$\delta J_{\alpha\dot{\alpha}} = \frac{i}{2} \partial_{\alpha\dot{\alpha}} \Lambda_0 \frac{\delta\Gamma}{\delta V} + \text{c.c.} \qquad (2.14)$$

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¿From this equation we can deduce the change (2.12) in $J_{\alpha\dot\alpha}$ required to obtain a gauge-invariant current.

¿From the Ward identity of equation (2.11) we can deduce the unintegrated superconformal WI by functionally differentiating with respect to L^{α}. It is:

$$D_{\alpha}\varphi\frac{\delta\Gamma}{\delta\varphi} - qD_{\alpha}\left(\varphi\frac{\delta\Gamma}{\delta\varphi}\right) + iW_{\alpha}\frac{\delta\Gamma}{\delta V} - i\bar{D}^{\dot\alpha}J_{\alpha\dot\alpha}\cdot\Gamma = -iD_{\alpha}S\cdot\Gamma \qquad (2.15)$$

We can derive this Ward identity by the following argument. Given that the identity contains only gauge invariant quantities it must involve only $\varphi, \bar\varphi, W_{\alpha}$ and S and Γ. The Ward identity consists of three types of term: terms that correspond to the variation of the fields, a term of the form $\bar{D}^{\dot\alpha}J_{\alpha\dot\alpha}\cdot\Gamma$ and a term with the anomaly S. The last two terms must be such that the anomaly equation (1.15) holds in Green's functions. This leaves only the first type of term which we can fix by using dimensional analysis and by demanding that the coefficients agree with those of the free theory.

We now give an alternative form of the Ward identity that includes the parameters of the superconformal transformations and involves the current J^{f} defined in section one. The advantage of this formulation is that one can make direct contact with the variation of the effective action under superconformal transformations and one can include all transformations in one Ward Identity. The transformation of φ and V under superconformal transformations are given by

$$\delta\varphi = X_{f}\varphi + q\triangle\varphi,$$
$$\delta V = X_{f}V, \qquad (2.17)$$

where X_{f} is a superconformal Killing vector (note that it simplifies when acting on chiral fields) and

$$\triangle = \partial_{\alpha\dot\alpha}f^{\alpha\dot\alpha} - D_{\alpha}\phi^{\alpha}. \qquad (2.18)$$

We can deduce the required form of the Ward identity by substituting

$$L^{\alpha} = f^{\alpha\dot\gamma}\bar{D}_{\dot\gamma}\delta^{8}(z-z') + \frac{1}{2}\bar{D}_{\dot\gamma}f^{\alpha\dot\gamma}\delta^{8}(z-z') \qquad (2.19)$$

into equation (2.11) to get

$$\delta\varphi\frac{\delta\Gamma}{\delta\varphi} + \left(f^{\alpha\dot\gamma}W_\alpha\bar D_{\dot\gamma} - \frac{1}{2}\bar D_{\dot\gamma}f^{\alpha\dot\gamma}W_\alpha\right)\frac{\delta\Gamma}{\delta V} - \bar D^2 J^f = \bar D^{\dot\alpha}f^\gamma_{\dot\alpha}D_\gamma S$$
$$- if^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}S + \left[\frac{1}{3}\partial_{\alpha\dot\alpha}\left(f^{\alpha\dot\alpha}\varphi\frac{\delta\Gamma}{\delta\varphi}\right) + \frac{i}{6}D_\alpha\left(\bar D_{\dot\alpha}f^{\alpha\dot\alpha}\varphi\frac{\delta\Gamma}{\delta\varphi}\right)\right], \qquad (2.20)$$

where $\delta\varphi$ is given in equation (2.16). We note that the final term in brackets on the right hand side is a total derivative.

Finally, we can find the integrated form of this equation by integrating over chiral superspace and adding the complex conjugate. The result is

$$\int d^4x d^2\theta\left\{\delta\varphi\frac{\delta\Gamma}{\delta\varphi} + \left(f^{\alpha\dot\gamma}W_\alpha\bar D_{\dot\gamma} - \frac{1}{2}\bar D_{\dot\gamma}f^{\alpha\dot\gamma}W_\alpha\right)\frac{\delta\Gamma}{\delta V}\right\} + \text{c.c.} = 2i\int d^4x d^2\theta\triangle S\cdot\Gamma + \text{c.c.}$$
$$(2.21)$$

where $\triangle$ is defined in equation (2.18). The current term does not contribute since $[D^2,\bar D^2]J^f$ is a total space-time derivative.

An alternative form of the anomalous Ward identity [18,19] can be given by including, in addition to the supergravity field $H^{\alpha\dot\alpha}$, the chiral compensator ϕ. In this case, equation (2.5) generalises to

$$2\int d^4x d^4\theta\; H^{\alpha\dot\alpha}J_{\alpha\dot\alpha} + \left\{2\int d^4x d^2\theta\phi S + \text{c.c}\right\}. \qquad (2.22)$$

The action of equation (2.1) plus the above term is invariant to zeroth order in the supergravity fields if we take $H^{\alpha\dot\alpha}$ to vary according to equation (2.6) and we take the variation of ϕ to be given by

$$\delta\phi = \frac{i}{2}\bar D^2 D_\alpha L^\alpha. \qquad (2.23)$$

Clearly, when the matter fields satisfy their equations of motion then the variation of the action implies that the supercurrent satisfies equation (1.15) from which we recognise S as the part of the effective action which is not superconformally invariant, that is the anomaly.

Upon varying the effective action plus equation (2.22) we can read off, as before, the coefficient of L^α to find the Ward identity with the anomaly automatically encoded. As such, we recover equations (2.15), (2.20) and (2.21).

The chiral compensator ϕ contains the component fields (4+4) required to complete the conformal supergravity multiplet of fields (8+8) to the old minimal Poincaré supergravity theory (12+12). As such, it necessarily couples to the chiral anomaly as shown in equation (2.22).

Clearly, we can keep the dependence of the effective action on the supergravity fields $H^{\alpha\dot\alpha}$ and ϕ and replace the presence of the current and the anomaly in the Ward identity by suitable functional derivatives with respect to the supergravity fields and then set the supergravity fields to zero. In particular, one can carry out this procedure for the anomaly alone.

3. Ward Identities for N=2 Supersymmetric Yang-Mills Theory

The $N = 2$ supersymmetric Yang-Mills theory [20] is described by a complex scalar superfield W which transforms under the adjoint representation of the gauge group. This superfield is covariantly chiral, i.e. $\bar\nabla^i_{\dot\alpha} W = 0$, and satisfies the constraint

$$\nabla_{ij} W = \bar\nabla_{ij}\bar W \tag{3.1}$$

where

$$\nabla_{ij} = \nabla_{\alpha(i}\nabla^\alpha_{j)}, \tag{3.2}$$

$\nabla_{\alpha i}$ is the spinorial covariant derivative including the gauge connection. The components of the superfield W are a complex scalar field, spinor fields in an $SU(2)$ doublet, the field strength tensor of the spacetime gauge field, and an $SU(2)$ triplet of auxiliary fields. We shall denote the superspace field strength tensor in the Abelian case by A.

The constraints (3.1) are solved in the Abelian case by

$$A = \bar{D}^4 D^{ij} V_{ij}, \tag{3.3}$$

where $V_{ij} = V_{ji}$ is the (unconstrained) superfield prepotential that contains the spacetime gauge potential [21]. The solution in the non-Abelian case is more complicated, but can still be written in terms of an unconstrained superfield V_{ij} [22]. Alternatively, one can use the harmonic superspace formalism [23] which allows one to use a prepotential of dimension zero, but we shall not consider this possibility here.

For the free theory, the action is given by

$$\int d^4x d^4\theta A^2 + \text{c.c.}, \tag{3.4}$$

and the supercurrent is given by

$$J = A\bar{A} \tag{3.5}$$

It is easy to check that it is conserved, i.e. $D_{ij} J = 0$, by virtue of the equation of motion $D_{ij} A = 0$.

We now derive the Ward identity for the $N = 2$ supersymmetric Yang-Mills theory. We could do this, as for the $N = 1$ case, by considering the variation of the action under super reparametrisations. However, the structure of $N = 2$ superspace supergravity is significantly more complicated than that of $N = 1$ supergravity and we shall not give the details of this approach here. Instead, we shall derive the identity heuristically using gauge invariance and dimensional analysis as we did for the $N = 1$ case. The Ward identity must again have the same three types of term and the ones involving the current and the anomaly must be consistent with the relation between the current J and the anomaly $\bar{S}$ of equation (1.16) in the sense that this latter equation is realised as an operator equation in Green's functions. The first type of terms which are associated with the variation of the effective action must contain $\dfrac{\delta\Gamma}{\delta V^{ij}}$, a function of A and possible covariant derivatives acting on these. We note that $\dfrac{\delta\Gamma}{\delta V^{ij}}$ has dimension two, but the dimension of all terms

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in the Ward identity is the same as that of the current term which is given by $D^{ij}J$ and has dimension 3. Taking all this information into account and fixing the one unknown coefficient by inserting the known current for the free classical theory we find that the Ward identity is given by

$$\bar{A}\frac{\delta\Gamma}{\delta V^{ij}} - D^{ij}J \cdot \Gamma = -\bar{D}^{ij}\bar{S} \cdot \Gamma \tag{3.6}$$

Using the expression for J_{ij} of equation (1.13) we find the Ward Identity which corresponds contains all the superconformal currents and their parameters; it is given by

$$-\frac{i}{4}\bar{D}_{\dot{\gamma}(i}f_\alpha{}^{\dot{\gamma}}\bar{A}\frac{\delta\Gamma}{\delta V^{jk)}} + D_{\alpha(i}J^f_{jk)} \cdot \Gamma = -\frac{i}{4}\bar{D}_{(jk}(\bar{D}_{\dot{\gamma}i)}f_\alpha{}^{\dot{\gamma}}\bar{S}) \cdot \Gamma. \tag{3.7}$$

The effective action for the U(1) field will contain two types of term, one of which is a full integral over superspace and is a function of A and $\bar{A}$ and the other which is an integral only over a chiral sub-integral of superspace and is a function of A only. We can write the latter contribution in the form

$$\Gamma_c = \int d^4x d^4\theta F(A) + \text{c.c.}. \tag{3.8}$$

The above definition of F differs from that in the literature. To recover the usual definition we should take $F \to -\frac{i}{16\pi}F$. The absence of the usual factors simplifies all the following equations. We now focus on the constraints imposed by the Ward identity on this latter contribution since this is the low energy effective action that appears in the Seiberg-Witten formalism. The variation of Γ_c with respect to V^{ij} is given by

$$\frac{\delta\Gamma_c}{\delta V^{ij}} = \bar{D}^{ij}\bar{F}' + D^{ij}F', \tag{3.9}$$

where $F' = \dfrac{\partial F}{\partial A}$.

Using the identity

$$\bar{A}\bar{D}_{jk}\bar{F}' = \bar{D}_{jk}(\bar{A}\bar{F}' - 2\bar{F}) + D_{jk}(A\bar{F}') \tag{3.10}$$

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we find that we can write the first term in the Ward identity of equation (3.6) when restricted to Γ_c as

$$\bar{A}\frac{\delta\Gamma_c}{\delta V^{ij}} = \bar{D}^{ij}(\bar{A}\bar{F}' - 2\bar{F}) + D^{ij}(A\bar{F}' + \bar{A}F') \tag{3.11}$$

Substituting this expression into equation (3.6) we find the Ward Identity can be written as

$$\bar{D}^{ij}(\bar{S} + \bar{A}\bar{F}' - 2\bar{F}) = -D^{ij}(J - A\bar{F}' - \bar{A}F') \tag{3.12}$$

We may rewrite this equation in the form

$$\bar{D}^{ij}\hat{J} = D^{ij}\hat{\bar{S}} \tag{3.13}$$

where

$$\hat{J} = J - A\bar{F}' - \bar{A}F', \quad \text{and} \quad \hat{\bar{S}} = \bar{S} + \bar{A}\bar{F}' - 2\bar{F} \tag{3.14}$$

We can regard $\hat{S}$ and $\hat{J}$ as a redefined anomaly and current respectively. This remarkable simplification of the Ward identity associated with the restricted effective action of equation (3.8) is essential for the derivation of the identity which is the subject of this paper. We now define a corresponding $\hat{J}^f_{ij}$ which is given by equation (1.13) except that we replace J with $\hat{J}$ so that

$$D_{\alpha(i}\hat{J}^f_{jk)} = -\frac{i}{4}\bar{D}_{\dot{\gamma}(i}f^{\dot{\gamma}}_\alpha D_{jk)}\hat{J} \tag{3.15}$$

Using equation (1.4), we find that

$$D_{\alpha(i}\hat{J}^f_{jk)} = -\frac{i}{4}(\bar{D}_{\dot{\gamma}(i}f_\alpha{}^{\dot{\gamma}})\bar{D}_{jk)}\hat{\bar{S}} = -\frac{i}{4}\bar{D}_{(jk}(\bar{D}_{\dot{\gamma}i)}f^{\dot{\gamma}}_\alpha\hat{\bar{S}}) \tag{3.16}$$

To obtain the integrated Ward identity we act with $\int d^4x \bar{D}^{ij}D^{\alpha k}$ on equation (3.16) and add the complex conjugate. The term involving $\hat{J}^f_{ij}$ does not contribute as it is a space-time derivative. This leaves

$$\int d^4x \bar{D}^{ij}D^{\alpha k}\bar{D}_{(jk}(\bar{D}_{\dot{\gamma}i)}f_\alpha{}^{\dot{\gamma}}\hat{\bar{S}}) + \text{c.c.}$$

$$= \int d^4x \bar{D}^{ij}\bar{D}_{(jk}\big[-i\delta^k_{i)}\partial_{\alpha\dot{\gamma}}f^{\alpha\gamma} + \bar{D}_{\dot{\gamma}i)}D^k_\alpha f^{\alpha\dot{\gamma}}\big]\hat{\bar{S}} + \text{c.c.} \tag{3.17}$$

$$= \frac{4i}{3}\int d^4x d^4\bar{\theta}\{\triangle\hat{\bar{S}}\} + \text{c.c.} = 0.$$

In the last step we have used the identities

$$\bar{D}^{ij}\bar{D}_{kl} = \frac{1}{6}(\delta_k^i \delta_l^j + \delta_k^j \delta_l^i)\bar{D}^{mn}\bar{D}_{mn}, \qquad \bar{D}_{\dot{\alpha}(k}\bar{D}_{ij)} = 0, \tag{3.18}$$

and the definitions

$$\int d^4\bar{\theta} = \bar{D}^{ij}\bar{D}_{ij}, \quad \triangle = \partial_{\alpha\dot{\alpha}}f^{\alpha\dot{\alpha}} + \bar{D}_{\dot{\gamma}i}\bar{\phi}^{\dot{\gamma}i}. \tag{3.19}$$

Rewriting (3.17) in terms of S using equation (3.14) and taking the complex conjugate, we find the final result

$$\int d^4x d^4\theta \triangle (A\frac{\partial F}{\partial A} - 2F + S) + \text{c.c} = 0 \tag{3.20}$$

Later in the paper it will also be useful to consider the $N = 2$ analogue of introducing the supergravity fields discussed for the case of $N = 1$ at the end of the last section. Of most significance to us will be the rôle of the supergravity compensator. The (24+24) set of fields of $N = 2$ superconformal supergravity can be compensated in a number of ways to form a (40+40) set of Poincaré supergravity fields. However, in this procedure one always adds a (8+8) chiral compensator which can be represented by a reduced chiral superfield ϕ_r subject to the constraints $\bar{D}_{\dot{\alpha}i}\phi_r = 0$, $D^{ij}\phi_r = \bar{D}^{ij}\phi_r$. For a review of this compensation mechanism we refer the reader to reference [29].

4. Application of the Ward Identity to Spontaneously Broken Yang-Mills Theory

The authors of reference [9] considered $N = 2$ $SU(2)$ Yang-Mills theory spontaneously broken to $U(1)$ and, assuming this theory to exhibit electromagnetic duality, were able to derive an expression for the low energy effective action. One way of defining their low energy effective action would be to regard it as that obtained by simply carrying out the functional integral for all the massive fields, but other definitions have been suggested. Whichever method is considered the low energy effective action is taken to be of the form of equation (3.8) and so depends holomorphically on only on the $N = 2$ Abelian superfield A.

In the discussions of reference [9] two possible variables are discussed for the formulation of the above effective action. The variable A is, as equation (3.8) makes clear, the one in terms of which we can write the effective action in a manifestly $N = 2$ supersymmetric manner. The other variable U is defined by $U = \frac{1}{2} \text{tr} \, W^2$. For the free and perturbative theory the relation between $a = \langle A \rangle$ and $u = \langle U \rangle = \frac{1}{2} \langle \text{tr} \, W^2 \rangle$ is simply $u = \frac{1}{2} a^2$. However, for the full non-perturbative theory the relation between a and u is complicated. The variable a viewed as a function of u contains singularities at isolated points. The determination of the monodromies around these singularities provides the mechanism [9] for determining a and $\dfrac{\partial F}{\partial a}$ as a function of u and hence F as a function of a.

When discussing the $N = 2$ superconformal transformations of the effective action we must realise the transformations in terms of the $N = 2$ superfield A since this is the variable which carries the standard representation of $N = 2$ supersymmetry. As a result, the superconformal Ward identity is given by equations (3.7) or (3.13) with the variable A as indicated.

The anomaly S must be single valued with respect to modular transformations around the singular points and must be a gauge-invariant object. Further, the anomaly S has dimension two and is a gauge-invariant chiral superfield. The only possible candidate is

$$S = \frac{c}{2} \text{tr} \, W^2 = cU, \tag{4.1}$$

where c is a constant. One can verify that the above S is consistent with the R transformations of $N = 2$ supersymmetry. The expectation value of the anomaly is therefore u. A more detailed analysis shows that although one can have several possible anomalies in the N=2 superfield current J, the only one possible for an $N = 2$ Yang-Mills theory is the chiral one above.

We noted that this anomaly is equivalent to the addition of a chiral multiplet which is non-reduced i.e. one which does not satisfy the analogue of equation (3.1). The anomaly S appears in the current equation (1.16) in the form $D^{ij} S \equiv L^{ij}$ and it is straight forward to

verify that this term satisfies the equations $D^{\alpha(i}L^{jk)} = 0 = \bar{D}^{\dot{\alpha}(i}L^{jk)}$ and so is a complex linear multiplet.

Taking these facts into account equation (3.20) can be written as

$$\int d^4x d^4\theta \triangle (A\frac{\partial F}{\partial A} - 2F + cU) + \text{c.c} = 0 \tag{4.2}$$

The chiral superfield $\triangle$ can be shown, using the constraints of equation (1.4), to be independent of space-time, but dependent on θ. The coefficients in the theta expansion are the arbitrary parameters of the conformal group of equation (1.8). While not all the coefficients of $\triangle$ are non-zero, equation (4.2) implies that the integral over a chiral superspace of $\triangle$ times a superfield, which is a function of A, vanishes for any chiral superfield A. As such, one can conclude that

$$A\frac{\partial F}{\partial A} - 2F = cU \tag{4.3}$$

plus a constant term and a term linear in A. These latter terms are not consistent with R symmetry and can also be discarded for reasons given later. Hence, if we consider the fields to be independent of space-time or taking the vacuum expectation value of the above equation we find that

$$a\frac{\partial F}{\partial a} - 2F = cu \tag{4.4}$$

We can determine the constant c by considering the large field limit in which perturbation theory is valid. In this regime $u = \frac{1}{2}a^2$ and after rescaling F, as discussed below equation (3.8), so as to agree with the rest of the literature we have $F = \frac{1}{2}\tau_{cl}a^2 + \frac{i}{2\pi}a^2 \ln \frac{a^2}{\Lambda^2}$. Hence we recognise that $c = 8\pi i\beta_1$ where β_1 is the coefficient of the one-loop β-function.

The above discussion generalises in a rather straightforward way to the case when the N=2 Yang-Mills theory has gauge group G spontaneously broken to $U(1)^r$ where r is the rank of G. The Ward identity of equation (3.6) is the same except that the first term is now $\bar{A}_n\frac{\delta\Gamma}{V_n^{ij}}$ where A_n and V_n^{ij}, $n = 1, \ldots r$ are the chiral superfield strength and prepotential respectively of the nth $U(1)$ factor. The Ward identity can be manipulated as before with the appropriate label and sum over n added to certain terms. The expectation value of

the anomaly is given by $< S >= 2c < TrW^2 >\equiv cu$ and so we find the constraint

$$\sum_{m=1}^{r} a_m \frac{\partial F}{\partial a_m} - 2F = 8\pi i\beta_1 u. \tag{4.5}$$

It is straightforward to rederive this equation in the presence of $N = 2$ matter and we hope to do carry this out elsewhere. Equation (4.5) is in agreement with references [15] and [16] where it was derived using the hypereliptic curve and the Whitham dynamics associated with the theory, but was only established for the gauge groups $SU(N)$, $SO(N)$and $Sp(N)$ with certain $N = 2$ matter.

In this case of a $N = 2$ theory with no superconformal anomaly we can repeat the above steps and then F will obey equation (4.3), but with no right hand side as $\beta_1 = 0$. This equation then determines that $F = dA^2$ where d is a constant. Hence in the case of the non-anomalous theories equation (4.4) determines the chiral part of the effective action. This is in agreement with the argument given in reference [13] that found that the Greens functions of the chiral sector of these theories are determined by superconformal invariance up to constants. In the case of N=2 Yang-Mills, one can readily show, by explicitly applying superconformal transformations to the Green's functions and demanding that the result vanish that only the two point Green's function is non-zero.

The appearance of an elliptic curve in the non-perturbative solution of Seiberg and Witten [9] for $N = 2$ Yang-Mills theory prompted the authors of references [25] to associate an integrable system with the non-perturbative solution. In particular, they identified the solution with Whitham dynamics. In reference [26], it was further shown that the Seiberg-Witten solution was only be identified with Whitham dynamics provided the function F obeyed the equation

$$a\frac{\partial F}{\partial a} - 2F = -\sum_n T_n \frac{\partial F}{\partial T_n}. \tag{4.6}$$

where the T_n, $n = 0, 1 \ldots$ are the times of the Whitham dynamics. It has been argued [15,16] that one can set $T_n = 0$, $n > 1$ and, in the presence of massless $N = 2$ matter,

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$T_0 = 0$, leaving only one the term containing T_1 on the right hand side. In the explicit examples studied [15,16], it has been found, using the explicit form of the hyperelliptic curve that this term does indeed equal $8\pi i u$ as required for agreement with equation (4.5). This equation has also been found [27] to play an important rôle in the derivation of N=2 Yang-Mills theory from superstring theories.

The alternative form of the condition of equation (4.6) also has a natural interpretation in terms of the derivation given in this paper. As pointed out at the end of section two we can, in $N = 1$ supersymmetric theories, replace the anomaly term by a suitable derivative with respect to the supergravity conformal compensating chiral superfield. For the case of $N = 2$ supersymmetric theories with the anomaly structure of equation (4.1), it is natural to take the conformal supergravity compensator to be a non-reduced chiral superfield which we also denoted by ϕ. The coupling between the anomaly and the compensator is then given by $\kappa \int d^4x d^4\theta \phi S$. We observe that adding this term to F and functionally differentiating with respect to ϕ and setting $\phi = 0$ contributes the term U in equation (4.3). In particular, differentiating with respect to the highest component, denoted t_1, of ϕ will result in u. Hence we can replace the anomaly in equation (4.3) and the term u in equation (4.5) by differentiation with respect to ϕ and t_1 respectively and as such, we can cast equation (4.5) in the form of equation (4.6) if we set $t_1 \propto \ln T_1$. The above non-reduced chiral compenstor includes the reduced chiral compensator ϕ_r, discussed at the end of section three that plays such a special rôle in the geometry of $N = 2$ couplings to supergravity. Presumeably, the other times T_n, $n > 1$ that appear in equation (4.6) are related to the components of other "compensating" superfields required to construct the superstring. We will return to a more detailed analysis of this point and the associated supergravity theory in a subsequent paper [28].

For the non-conformal theory, equation (4.4) does not determine the chiral effective action of equation (3.8) completely. In particular, one is missing the knowledge of u as a function of A which, if provided, would allow one to solve to write the above differential equation in terms of only one variable. Very roughly speaking we are missing about half

the information contained in the solution. Seen from the perspective of reference [26], we are missing the Whitham dynamics itself. It would be interesting to also give a derivation of this missing information from the basic theory.

Some papers [16,26] have observed that the constraint of equation (4.6) looks similar to the L_0 constraint found in matrix models. It is thought to be true in matrix models that the entire solution is given by the imposition of all of the positive Virasoro constraints, i.e. L_n, $n \geq 0$. One might also hope that a corresponding statement is true for the $N = 2$ Yang-Mills theory and, in view of the work of the current paper, one may wonder if the constraints L_n, $n > 0$ correspond to Ward Identities for some additional, possibly broken, symmetries of the theory.

It would also be interesting to attempt to obtain information about $N = 1$ supersymmetric theories using arguments analogous to those used in section four, but starting from equation (2.21).

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